Problem 1: $\mathbb{E}\,\|A\Omega B\|_F^2 = \|A\|_F^2\,\|B\|_F^2$, for Ω Gaussian, A, B fixed

Gaussian matrix & Frobenius norm – invariant under orthogonal transformations

$\Rightarrow$ assume A and B diagonal

$$\Rightarrow \mathbb{E}\,\|A\Omega B\|_F^2 = \mathbb{E}\left[\sum_{ij} |a_{ii}\,\Omega_{ij}\,b_{jj}|^2\right] =$$

$$= \sum_{ij} |a_{ii}|^2 |b_{jj}|^2 = \|A\|_F^2\,\|B\|_F^2$$

Problem 2:

1) A $m \times n$ $\quad \Omega$ $n \times k$, $\quad k \leq \min\{m, n\}$

1. $\underset{m\times n}{Y} = \underset{n\times k}{A\Omega}$ $\quad O(mnk)$

2. $\underset{m\times k}{Y} = QR$ $\quad O(mk^2)$

3. $\underset{k\times m}{B} = \underset{k\times m}{Q^T}\,\underset{m\times n}{A}$ $\quad O(mnk)$

4. $\underset{k\times n}{B} = \underset{k\times k}{\tilde{U}}\Sigma V^T$ $\quad O(nk^2)$

5. $\underset{m\times k}{U} = \underset{m\times k}{Q}\,\underset{k\times k}{\tilde{U}}$ $\quad O(mk^2)$

$\longrightarrow O(mnk)$

2) $A \approx QQ^T A QQ^T \qquad A = A^T$

$\|A - QQ^T A\|_2 \leq \varepsilon$

$$\|A - QQ^T A QQ^T\|_2 = \|A - QQ^T A + QQ^T A - QQ^T A QQ^T\|_2$$

$$\leq \underbrace{\|A - QQ^T A\|_2}_{\leq \varepsilon} + \|QQ^T(A - AQQ^T)\|_2$$

$$\leq \varepsilon + \|Q^T A(I - QQ^T)\|_2 =$$

$$= \varepsilon + \|(I - QQ^T)\underbrace{A^T}_{=A} Q\|_2 = \varepsilon + \|(I - QQ^T)AQQ^T\|_2$$

$$\leq \varepsilon + \underbrace{\|(I - QQ^T)A\|_2}_{\leq \varepsilon} \underbrace{\|QQ^T\|_2}_{=1} \leq 2\varepsilon$$

3) $A \approx Q\underbrace{Q^T A Q}Q^T$

$B = Q^T A Q \quad \rightarrow \quad B = B^T$

$B = \tilde{U} \Lambda \tilde{U}^T$ (eigendecomposition)

$$\Rightarrow A \approx \underbrace{Q\tilde{U}}_{:=U} \Lambda \underbrace{(Q\tilde{U})^T}_{U} = U \Lambda U^T$$

Problem 3 : P orthogonal projector $\rightarrow$ $P^2 = P$, $P^T = P$

D non-negative diagonal

$t > 1$

$$(*) \quad \|PDP\|_2^{\,t} \leq \|PD^tP\|_2$$

PROVE: $\|PA\|_2 \leq \|P(AA^T)^{\ell}A\|_2^{\frac{1}{2\ell+1}}$, $\ell \in \mathbb{N}$

$$\underset{m\times n}{A} = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,V^T$$

$$\|PA\|_2^{2(2\ell+1)} = \|PAA^TP\|_2^{2\ell+1} = \|(U^TPU)\Sigma V^TV\Sigma(U^TPU)\|_2^{2\ell+1}$$

$$= \|(U^TPU)\Sigma^2(U^TPU)\|_2^{2\ell+1} \overset{(*)}{\leq} \|(U^TPU)\Sigma^{2(2\ell+1)}(U^TPU)\|_2$$

$$= \|P(AA^T)^{2\ell+1}P\|_2 = \|\underbrace{P(AA^T)^{\ell}}\underbrace{(AA^T)(AA^T)^{\ell}P}\|_2$$

$$= \|P(AA^T)^{\ell}A\|_2^2$$

$$\Rightarrow \|PA\|_2 \leq \|P(AA^T)^{\ell}A\|_2^{\frac{1}{2\ell+1}}$$

Problem 4 : $C \in \mathbb{R}^{m \times n}$

$w_i \in \mathbb{R}^n$, $i = 1, 2, \dots, r$ Gaussian vectors

$\|C\|_2 \leq 10 \sqrt{\frac{2}{\pi}} \max_{i=1,\dots,r} \|Cw_i\|_2$ holds with probability $1 - 10^{-r}$.

- for $Z \sim \chi_1^2$, $\mathbb{P}(Z < \mu) \leq \sqrt{\frac{2}{\pi}} \sqrt{\mu}$, $\mu > 0$ (*)

- PROVE : x Gaussia, B pos. semi-def. , $\lambda_1 = \lambda_{max}(B)$

$$\mathbb{P}\left(\Theta^2 x^T B x < \lambda_1\right) \leq \sqrt{\frac{2}{\pi}}\, \Theta^{-1}$$

B symmetric $\Rightarrow$ $\underset{n \times n}{B} = U \Lambda U^T$, $U = [u_1 \dots u_n]$ eigenvectors $\rightarrow$ form ONB

$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$x \sim N(0, I_n)$ $\rightarrow x = \sum_{i=1}^{n} x_i u_i$

$$\mathbb{P}\left(\Theta^2 x^T B x < \lambda_{max}\right) = \mathbb{P}\left(\Theta^2 \sum_{i=1}^{n} \lambda_i x_i^2 < \lambda_{max}\right)$$

B pos. semi-def. $\rightarrow$ $\lambda_{max} \geq \dots \geq \lambda_{min} \geq 0$ $\Rightarrow$

$$\mathbb{P}\left(\Theta^2 \underbrace{\sum_{i=1}^{n} \underset{\geq 0}{\lambda_i} \underset{\geq 0}{x_i^2}}_{\geq \lambda_{max} \cdot x_1^2} < \lambda_{max}\right) \leq \mathbb{P}\left(\Theta^2 \lambda_{max} x_1^2 < \lambda_{max}\right) = \mathbb{P}\left(\Theta^2 x_1^2 < 1\right)$$

(x_1 eigenvector corresponding to λ_{max})

$x = (x_1, \dots, x_n) \sim \mathcal{N}(0, I_n)$ $\Rightarrow$ $x_i^2 \overset{iid}{\sim} \chi_1^2$,

$$\mathbb{P}\left(\Theta^2 x_1^2 < 1\right) = \mathbb{P}\left(x_1^2 < \frac{1}{\Theta^2}\right) \overset{(*)}{\leq} \sqrt{\frac{2}{\pi}} \cdot \sqrt{\Theta^{-2}} = \sqrt{\frac{2}{\pi}} \cdot \Theta^{-1}$$

- A $m \times n$, x $m \times 1$

$$\mathbb{P}\left(\sqrt{\frac{2}{\pi}}\,\alpha\,\|Ax\|_2 < \|A\|_2\right) \le \alpha^{-1}$$

$$\mathbb{P}\left(\sqrt{\frac{2}{\pi}}\,\alpha\,\|Ax\|_2 < \|A\|_2\right) = \mathbb{P}\left(\left(\sqrt{\frac{2}{\pi}}\alpha\right)^2 \|Ax\|_2^2 < \|A\|_2^2\right)$$

$$= \mathbb{P}\left(\left(\sqrt{\frac{2}{\pi}}\alpha\right)^2 x^t A^t A x < \lambda_{max}(A^t A)\right)$$

$\Theta := \sqrt{\frac{2}{\pi}}\alpha$, $B := A^t A$ pos. semidefinite

PREVIOUS STEP $\Longrightarrow$

$$\le \sqrt{\frac{2}{\pi}} \cdot \left(\sqrt{\frac{2}{\pi}}\alpha\right)^{-1} = \alpha^{-1}$$

- $$\Rightarrow \mathbb{P}\left(\|C\|_2 \le 10\sqrt{\frac{2}{\pi}} \max_{i=1,\dots,r} \|Cw_i\|_2\right) =$$

$$= 1 - \mathbb{P}\left(10\sqrt{\frac{2}{\pi}} \max_{i=1,\dots,r} \|Cw_i\|_2 < \|C\|_2\right)$$

$$= 1 - \mathbb{P}\left(\bigcap_{i=1}^{r} \left\{10\sqrt{\frac{2}{\pi}}\,\|Cw_i\|_2 < \|C\|_2\right\}\right)$$

$$= 1 - \mathbb{P}\left(10\sqrt{\frac{2}{\pi}}\,\|Cw_i\|_2 < \|C\|_2\right)^r$$

$$\ge 1 - 10^{-r}$$

Problem 6: $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$, $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times k}$, $C \in \mathbb{R}^{l \times n}$

$$\min_{A_r} \left\{ \left\| \begin{bmatrix} A-A_r & B \\ C & 0 \end{bmatrix} \right\| : \operatorname{rank}(A_r) \leq r \right\}$$

1) $\|\cdot\| \equiv \|\cdot\|_F$

$$\left\| \begin{bmatrix} A-A_r & B \\ C & 0 \end{bmatrix} \right\|_F^2 = \|A-A_r\|_F^2 + \|B\|_F^2 + \|C\|_F^2 \longrightarrow \text{minimal when } A_r = J_r(A)$$

2) $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$r = 1$

- $A_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \left\| \begin{bmatrix} A-A_1 & B \\ C & 0 \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\|_2 = 1$, $\|A-A_1\|_2 = 1$

- $J_1(A) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \|A - J_1(A)\|_2 = 0.5$

but: $\left\| \begin{bmatrix} A-J_1(A) & B \\ C & 0 \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} 0.5 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\|_2 = \sqrt{1+0.5^2} > 1$

Problem 5.

```
clear all

n=100;
x=zeros(n);
for i=1:n
    x(i)=i/10;
end
A=zeros(n,n);
for i=1:n
    for j=1:n
        A(i,j)=1./(x(i)+x(j));
    end
end
B=zeros(n,n);
for i=1:n
    for j=1:n
        B(i,j)=1./((x(i)+x(j))^0.5);
    end
end

rA=rank(A)
rB=rank(B)

[Ua,Sa,Va]=svd(A,'econ');
[Ub,Sb,Vb]=svd(B,'econ');
Ua=Ua(:,1:rA);
Sa=Sa(1:rA,1:rA);
Va=Va(:,1:rA);
Ub=Ub(:,1:rB);
Sb=Sb(1:rB,1:rB);
Vb=Vb(:,1:rB);

C=A+B;
r=rank(C)

k=r+0;
Omega=randn(n,k);
Ya=Ua*(repmat(diag(Sa),1,k).*(Va'*Omega));
Yb=Ub*(repmat(diag(Sb),1,k).*(Vb'*Omega));

Yc=Ya+Yb;
[Q,R]=qr(Yc,0);
Ba=Q'*Ua*(repmat(diag(Sa),1,n).*Va');
Bb=Q'*Ub*(repmat(diag(Sb),1,n).*Vb');
B=Ba+Bb;
[U,S,V]=svd(B,'econ');
U=Q*U(:,1:r);
S=S(1:r,1:r);
V=V(:,1:r);

norm(C-U*S*V')
sv=svds(C,r+1);
sv(end)
```