Problem 1. Prove that the volume of a square matrix is equal to the product of its singular values.

Problem 2. For $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times n}$, with $m > n$, let $A_i$ be a $n \times (m - 1)$ submatrix of $A$ without $i$-th column, and $B_i$ be $(m - 1) \times n$ submatrix of $B$ without $i$-th row. Prove that it holds that
$$\det(AB) = \frac{1}{m-n} \sum_{i=1}^{m} \det(A_iB_i).$$

Problem 3. This exercise is concerned with the lemma from slide 4 of Lecture 4.

1. Show that the inequality of the lemma cannot be improved in general by constructing a matrix $U = \begin{pmatrix} I_r \\ B \end{pmatrix}$ with $\max |b_{ij}| \leq 1$ and $\|B\|_2 = \sqrt{(n-r)r}$.

2. Develop an extension of the result of the lemma. Prove that for an arbitrary $n \times r$ matrix $U$ of rank $r$, there is an index set $I$ with $\#I = r$ such that
$$\frac{1}{\sigma_{\min}(U(I,:))} \leq \sqrt{r(n-r)} + \frac{1}{\sigma_{\min}(U)}.$$

Problem 4. For the matrix $Z_n$ defined on Slide 10, prove $(Z_n^{-1})_{ij} = 2^{j-i-1}$ for $j > i$.

Problem 5. Prove that any projector $\Pi \notin \{0, I\}$ satisfies $\|I - \Pi\|_2 = \|\Pi\|_2$.

Problem 6. Quite likely, this problem is very hard. Let $A \in \mathbb{R}^{n \times n}$ be symmetric positive definite and let $k \in [1, n]$. Prove or give a counterexample for the conjecture
$$\max\{|\det(A(I,J))| : I, J \subset \{1, \ldots, n\}, \#I = \#J = k\} = \max\{|\det(A(I,I))| : I \subset \{1, \ldots, n\}, \#I = k\}.$$

In particular, this would imply that there is a symmetric positive definite $k \times k$ submatrix of maximal volume.

Problem 7. MATLAB exercise.

1. Implement the greedy method from Slide 9.

2. Given an $m \times n$ matrix $A$ and an integer $r \leq \min\{m, n\}$, implement the following method for obtaining a rank-$r$ approximation:
   - Apply $r$ steps of the greedy method to left singular vectors $U_r$ of $A$ to determine $I$.
   - Apply $r$ steps of the greedy method to right singular vectors $V_r$ of $A$ to determine $J$.
   - Return rank-$r$ approximation $A(:, J)(A(I,J))^{-1}A(I,:)$.

Apply your implementation to the two matrices from Slide 34 for $r = 1, \ldots, 30$ (for Hilbert matrix) and $r = 1, \ldots, 100$ (for exponential). Plot the singular values and the obtained approximation error in the spectral norm.