**EXERCISE 2 – Low-rank approximation techniques**

**Problem 1.** Let \( A \in \mathbb{R}^{m \times n} \) with \( m \geq n \) and \( k \leq n \). Use von Neumann’s trace inequality to prove that
\[
\max \{ \| Q^T A \|_F : Q \in \mathbb{R}^{n \times k}, \ Q^T Q = I_k \} = \sqrt{\sigma_1^2 + \cdots + \sigma_k^2},
\]
where \( \sigma_1, \ldots, \sigma_k \) are the singular values of \( A \).

**Problem 2.**
1. Prove the basic properties of the angle between vectors stated on Slide 12 of the slides of Lecture 2.
2. Prove the basic properties of the angle between a vector and a subspace stated on Slide 13 of the slides of Lecture 2.

**Problem 3.** The goal of this exercise is to prove the projector characterization
\[
\sin \theta_1(\mathcal{X}, \mathcal{Y}) = \|XX^T - YY^T\|_2
\]
for orthonormal bases \( X, Y \) of \( \mathcal{X}, \mathcal{Y} \).

1. Explain why one can assume w.l.o.g that \( X = \begin{bmatrix} I_k & 0 \end{bmatrix} \) (Hint: Utilize the QR decomposition.)
2. Assuming that \( X = \begin{bmatrix} I_k & 0 \end{bmatrix} \), partition \( Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \) with \( Y_1 \in \mathbb{R}^{k \times k} \). Setting \( P = XX^T - YY^T \), show that \( P^2 \) is block diagonal. What are the diagonal blocks?
3. Using Point 2, compute the largest singular value of \( P \) in terms of the singular values of \( X^T Y \) and establish
\[
\sin \theta_1(\mathcal{X}, \mathcal{Y}) = \|XX^T - YY^T\|_2.
\]

**Problem 4.**
A function \( f : \mathbb{R}^{m \times n} \to \mathbb{R} \) is called lower semi-continuous (upper semi-continuous) at \( x_0 \in \mathbb{R}^{m \times n} \) if for every \( \epsilon > 0 \) there exists a neighborhood \( U \) of \( x_0 \) such that
\[
f(x) \geq f(x_0) - \epsilon \quad (f(x) \leq f(x_0) + \epsilon) \quad \text{for all} \quad x \in U.
\]

1. Prove that the rank function \( A \mapsto \text{rank}(A) \) is lower semi-continuous at every matrix \( A_0 \in \mathbb{R}^{m \times n} \). (Hint: Use the stability of singular values.)
2. Construct an example to show that the rank function is in general not upper semi-continuous.

**Problem 5.** The goal of this exercise is to recall the QR decomposition and illustrate its use in low-rank approximation.

Let \( X \in \mathbb{R}^{m \times n} \) with \( m \geq n \). Then there is an orthogonal matrix \( Q \in \mathbb{R}^{m \times m} \) such that
\[
X = QR, \quad \text{with} \quad R = \begin{pmatrix} R_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \ast \\ 0 \end{pmatrix},
\]
that is, \( R_1 \in \mathbb{R}^{n \times n} \) is an upper triangular matrix. In practice, one often uses the economy-size QR decomposition instead: Letting \( Q_1 \in \mathbb{R}^{m \times n} \) contain the first \( n \) columns of \( Q \), one obtains
\[
X = Q_1 R_1 = Q_1 \cdot \begin{pmatrix} \ast \\ 0 \end{pmatrix}.
\]

The computation of such an economy-size QR decomposition requires \( O(mn^2) \) operations.

1. Given \( A \in \mathbb{R}^{n \times n} \), partition \( A = [a_1, a_2, \ldots, a_n] \) with \( a_i \in \mathbb{R}^n \). Using the QR decomposition, show Hadamard’s inequality:
\[
|\det(A)| \leq \|a_1\|_2 \cdot \|a_2\|_2 \cdots \|a_n\|_2.
\]

Characterize the set of all \( n \times n \) matrices \( A \) for which equality holds.
2. Let $P \in \mathbb{R}^{m \times R}$, $Q \in \mathbb{R}^{n \times R}$, with $R \leq n \leq m$, be matrices with orthonormal columns. Let $A = PSQ^T$ and $r < R$. Prove that $P \cdot T_r(S) \cdot Q^T$ is a best rank-$r$ approximation of $A$ in the sense that

$$\|A - P \cdot T_r(S) \cdot Q^T\|_F = \|A - T_r(A)\|_F.$$ 

3. Using the result from Point 2, develop an algorithm of complexity $O(mR^2 + nR^2)$ for performing rank-$r$ truncation of a matrix $BC^T$ with $B \in \mathbb{R}^{m \times R}$, $C \in \mathbb{R}^{n \times R}$, $R \leq n \leq m$. Implement and test this algorithm in MATLAB.

4. Using the algorithm from Point 3, develop an algorithm of complexity $O(mr^2 + nr^2)$ for recompressing the sum of two rank-$r$ matrices back to rank $r$. Implement and test this algorithm in MATLAB.