ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

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1 ► Order criteria

Consider an explicit two-stage Runge-Kutta method given by the Butcher table

$$\begin{array}{c|cc} 0 & 0 & 0 \\ c & a & 0 \\ \hline & b_1 & b_2 \end{array}$$

Write down equations for the parameters such that the method is consistent of order two. Show that the order of consistency cannot be higher.

This Runge-Kutta method reads

$$y_1(t_0+h) = y_0 + hb_1 f(t_0, y_0) + hb_2 f(t_0 + hc, y_0 + haf(t_0, y_0))$$

let's choose $\boxed{a = c}$, otherwise it will be difficult. (cf. Lemma 2.7)

Then $y_0 + haf(t_0, y_0) = y_0 + hc\, y'(t_0) = y(t_0 + hc) + \mathcal{O}(h^2)$, so that, if f is sufficiently smooth

$$\begin{aligned} hb_2 f(t_0+hc, y_0 + haf(t_0,y_0)) &= hb_2\big(f(t_0+hc, y(t_0+hc)) + \mathcal{O}(h^2)\big) \\ &= hb_2\, y'(t_0+hc) + \mathcal{O}(h^3) \\ &= hb_2 y'(t_0) + h^2 b_2 c\, y''(t_0) + \mathcal{O}(h^3). \end{aligned}$$

Hence

$$y_1(t_0+h) = y_0 + (b_1+b_2)y'(t_0)h + b_2 c\, y''(t_0)h^2 + \mathcal{O}(h^3).$$

The terms up to order two should equal the ones in the Taylor expansion of y, that is,

$$\boxed{b_2 c = \tfrac{1}{2}}, \quad \boxed{b_1 + b_2 = 1}$$

For $a = c = \frac{1}{2}$ we get $b_2 = 1$, $b_1 = 0$, i.e. Runge's method.

<u>Consistency of order three is impossible</u>

Argument is like in Problem 1b): For the IVP

$$y' = y, \quad y(t_0) = y(0)$$

a two-step method will only produce a polynomial of degree two. ☐

2 ► Simpler order conditions for simple ODEs

Let $(A, \mathbf{b}, \mathbf{c})$ be an explicit s-stage Runge Kutta method, $s \geq p$, which is invariant under autonomization, i.e., $\mathbf{c} = A\mathbf{e}$.

(a) Consider the simplest autonomous initial value problem

$$\mathbf{y}'(t) = B\mathbf{y}(t), \quad \mathbf{y}(t_0) = \mathbf{y}_0 \in \mathbb{R}^d, \quad B \in \mathbb{R}^{d\times d}.$$

Show that the Runge-Kutta method has consistency order p if and only if

$$\mathbf{b}^\mathsf{T} A^{(\beta)} = \frac{1}{(\#\beta)!} \quad \text{for all linear trees } \beta = [[\cdots[\odot]\cdots]] \text{ with } \#\beta \leq p.$$

a) $y' = By \quad , \quad y(t_0) = y_0$

In this case $f(y) = By$, so $f^{(k)}(y) = 0$ if $k \geq 2$.
Therefore $f^{(\beta)} = 0$ if β contain any branching.
Comparing the Taylor expansion of lemmas 2.10 and 2.11 it follows that we need

$$b^T A^{(\beta)} = \frac{1}{\beta!} \quad \text{for all linear trees } [\cdots[\odot]\cdots] \text{ with } \#\beta \leq p.$$

For such trees $\beta! = (\#\beta)!$. ☑

(b) Consider the quadrature problem

$$\mathbf{y}'(t) = f(t), \quad \mathbf{y}(t_0) = \mathbf{y}_0 \in \mathbb{R}^d$$

of a function $f \in C^\infty([0,T])$. Show that the Runge-Kutta method has consistency order p if and only if

$$\mathbf{b}^\mathsf{T} A^{(\beta)} = \frac{1}{\#\beta} \quad \text{for all one-level trees } \beta = [\odot, \ldots, \odot] \text{ with } \#\beta \leq p.$$

b) $y' = f(t) \quad , \quad y(t_0) = y_0$

The numerical solution satisfies (component-wise)

$$y_1(t_0+h) = y_0 + h \sum_{i=1}^{s} b_i f(t_0 + c_i h)$$
$$= y_0 + h \sum_{i=1}^{s} b_i \left(\sum_{k=1}^{p} \frac{y^{(k)}(t_0)}{(k-1)!} (c_i h)^{k-1} + \mathcal{O}(|h|^p) \right)$$
$$= y_0 + \sum_{k=1}^{p} \frac{y^{(k)}(t_0)}{(k-1)!} \left(\sum_{i=1}^{s} b_i c_i^{k-1} \right) h^k + \mathcal{O}(|h|^{p+1}).$$

This leads to the condition

$$\sum_{i=1}^{s} b_i c_i^{k-1} = \frac{1}{k} \quad \text{for } k = 1, \ldots, p.$$

But for one-level trees $\beta = [\odot, \ldots, \odot]$ with $\#\beta = k \leq p$ we have $A_i^{(\beta)} = (Ae)_i^{k-1} = (\sum_j a_{ij})^{k-1} = c_i^{k-1}$. Therefore we could write the condition as

$$b^T A^{(\beta)} = \frac{1}{\#\beta} \quad \text{for all } \beta = [\odot, \ldots, \odot] \text{ with } \#\beta \leq p.$$

☑

3 ▶ Local vs. global error

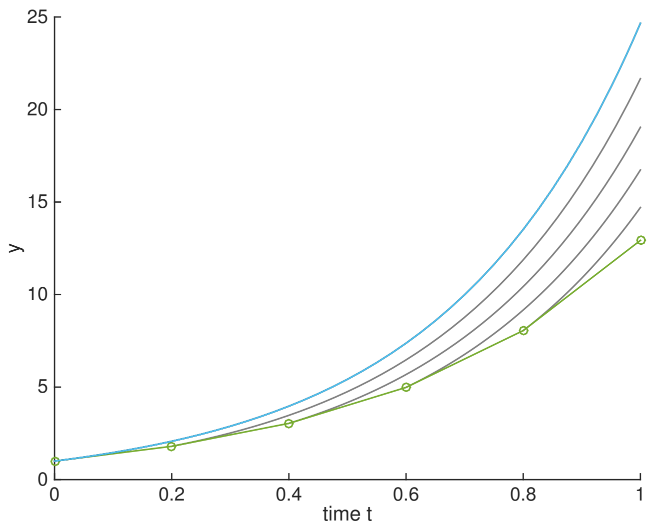
In this exercise, we will investigate the difference between the local and the global error. We consider the ODE

$$y'(t) = 1 - t + 3y, \quad y(t_0) = y_0$$

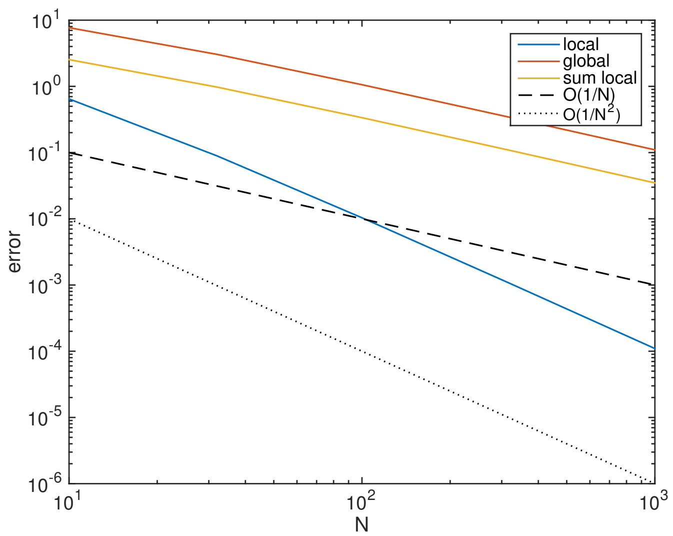
with exact solution

$$y(t) = Ce^{3t} - \frac{2}{9} + \frac{t}{3}, \quad C(t_0, y_0) = \frac{y_0 - \frac{t_0}{3} + \frac{2}{9}}{e^{3t_0}}.$$

1. Create a plot similiar to Figure 2.1 *(Lady Windemere's fan)* using an explicit Euler method on the interval $T = [0, 1]$ with stepsize $h = 0.2$ and initial condition $y(0) = 1$. To do this, you have to calculate the exact solutions $y(t)$ with initial conditions $y(t_i) = y_i$ obtained from the ith step of the explicit Euler scheme.



2. Plot the global error $|y(1) - y_N|$, where y_N is the last step of the Euler scheme and compare it to the maximum local error, both as functions of the stepsize h. What is the convergence order that you obtain for the local and the global error?



```
function windemere()

f = @(t,y) 1 - t + 3*y;

sol = @(t)  1/9 * (3*t + 11*exp(3*t) - 2);
T = [0,1];
y0 = 1;
[t,y] = ode45(f, T, y0);
[t5,y5] = expeuler( f, T, y0, 5);

hold on
for i = 2:5
    [tt,yy] = ode45(f, [t5(i),1], y5(i));
    plot(tt,yy,'-','Color',[0.5,0.5,0.5])
end

set(0,'defaultlinelinewidth',1.2)
set(gca,'fontsize',14)
plot(t5,y5,'-o')

xlabel('time␣t')
ylabel('y')
plot(t,y,'-k')
plot(t,sol(t))

ylim([0,25])

solC = @(t,C)  C*exp(3*t) - 2/9 + t/3;
N = round(logspace(1,3,5));
for i=1:5
    [t, y] = expeuler( f, T, y0, N(i) );
    y_local = zeros(length(t)-2,1);

    for j=2:length(t)-1
        C = (y(j) - t(j)/3 + 2/9 )/exp(3*t(j));
        y_local(j-1) = solC(t(j+1), C);
    end

    yex = sol(t);
    err_local(i) = max(abs(y(3:end) - y_local));
    err_global(i) = abs(yex(end) - y(end));
    err_sum_local(i) = sum(abs(y(3:end) - y_local));
end

figure
loglog( N, err_local)
set(gca,'fontsize',14)
hold on
loglog( N, err_global)
loglog( N, err_sum_local)

loglog( N, 1./N, '--k')
loglog( N, 1./(N.^2), ':k')
xlabel('N')
ylabel('error')
legend('local','global','sum␣local', 'O(1/N)', 'O(1/N^2)')
hold off
end


function [t, y] = expeuler( f, T, y0, N );
    t = linspace( T(1), T(2), N+1 )';
    h = t(2) - t(1);
    y = zeros( N+1, 1 );
    y(1) = y0;
    for i = 1:N
        y(i+1) = y(i) + h * f(t(i), y(i));
    end
end
```

4 ► Order isn't everything[1]

Solve the initial value problem

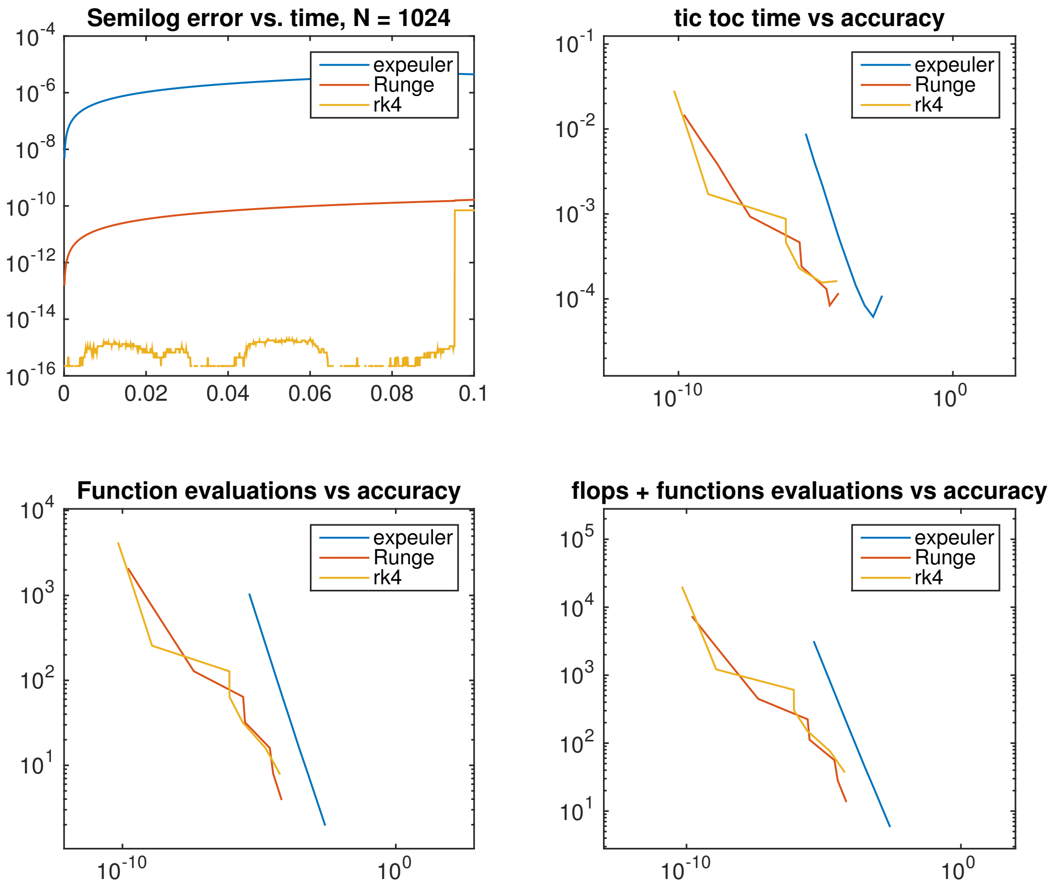
$$y'(t) = |1.1 - y| + 1, \quad t \in [0, 0.1], \quad y(0) = 1,$$

both analytically, and numerically using the explicit Euler method, Runge's method, and the classical RK4 method. Plot errors of all three methods against time.

Then write a script which does the same for various step sizes $N = 2^k$, $k = 1, 2, \dots, 10$, and measures the runtime of each method (use the commands `tic` and `toc`). For all three methods, plot the runtime versus the achieved final accuracy at $T = 0.1$ (use a double logarithmic scale). You may compare to a plot of the required number of function evaluations versus the final achieved accuracy.

Hint. The exact solution is

$$y(t) = \begin{cases} -1.1e^{-t} + 2.1, & 0 \le t \le \ln 1.1, \\ \frac{10}{11}e^{t} + 0.1, & \ln 1.1 \le t \le 0.1. \end{cases}$$



[1] Title and exercise taken from Deuflhard, Bornemann: Scientific Computing with Ordinary Differential Equations, Springer, 2002.

```matlab
% Exercise 4 Problem 3
% ===================

function ex4problem3
close all;

% set parameters
f = @(t,y) abs(1.1 - y) + 1;
T = 0.1;
tspan = [0,T];
y0 = 1;
N = 1024;

% exact solution
y = @(t) (2.1-1.1*exp(-t)).*(t<log(1.1)) + (0.1+10/11*exp(t)).*(t>=log(1.1));

% run expeuler, runge, and RK4
[t1,y1] = expeuler(f,tspan,y0,N);
[t2,y2] = runge(f,tspan,y0,N);
[t3,y3] = rk4(f,tspan,y0,N);

% errors
e1 = abs(y(t1) - y1);
e2 = abs(y(t2) - y2);
e3 = abs(y(t3) - y3);

% plot all in one figure
scrsz = get(0,'ScreenSize');
figure('Position',[1 scrsz(4)/2-100 scrsz(3)/2 scrsz(4)/2])

subplot(2,2,1)
semilogy(t1,e1,t2,e2,t3,e3);
legend('expeuler','Runge','rk4')
title(sprintf('Semilog error vs. time, N = %i',N))

% performance test
for i=1:1:10
    N = 2^i;

    % exp euler
    tic;
    [t1,y1] = expeuler(f,tspan,y0,N);
    runtime1(i) = toc;
    funeval1(i) = N;
    flops1(i) = 3*N;
    final_error1(i) = abs(y(T) - y1(N + 1));

    % Runge
    tic;
    [t2,y2] = runge(f,tspan,y0,N);
    runtime2(i) = toc;
    funeval2(i) = 2*N;
    flops2(i) = 7*N;
    final_error2(i) = abs(y(T) - y2(N + 1));

    % RK4
    tic;
    [t3,y3] = rk4(f,tspan,y0,N);
    runtime3(i) = toc;
    funeval3(i) = 4*N;
    flops3(i) = 19*N;
    final_error3(i) = abs(y(T) - y3(N + 1));
end

subplot(2,2,2)
loglog(final_error1,runtime1,final_error2,runtime2,final_error3,runtime3);
legend('expeuler','Runge','rk4')
title('tic toc time vs accuracy')
```

```
subplot(2,2,3)
loglog(final_error1,funeval1,final_error2,funeval2,final_error3,funeval3);
legend('expeuler','Runge','rk4')
title('Function evaluations vs accuracy')

subplot(2,2,4)
loglog(final_error1,flops1,final_error2,flops2,final_error3,flops3);
legend('expeuler','Runge','rk4')
title('flops + functions evaluations vs accuracy')

end

% Required functions
% ------------------

% Explicit Euler

function [t,y] = expeuler(f,tspan,y0,N)
    y(:,1) = y0;
    h = (tspan(2)-tspan(1))/N;
    t = tspan(1)*ones(1,N+1) + h*(0:N);
    for i =2:N+1
        y(:,i) = y(:,i-1) + h*f(t(i-1),y(:,i-1));
    end
end

% Runge's method

function [t,y] = runge(f,tspan,y0,N)
    y(:,1) = y0;
    h = (tspan(2)-tspan(1))/N;
    t = tspan(1)*ones(1,N+1) + h*(0:N);
    for i =2:N+1
        k1 = f( t(i-1), y(:,i-1) );
        k2 = f(t(i-1) + h/2, y(:,i-1) + h/2*k1);

        y(:,i) = y(:,i-1) + h*k2;
    end
end

% Classical RK4

function [t,y] = rk4(f,tspan,y0,N)

h = (tspan(2)-tspan(1))/N;
t = tspan(1)*ones(1,N+1) + h*(0:N);

t(1) = tspan(1);
y(:,1) = y0;

g=[1/6 1/3 1/3 1/6];

for i=2:N+1
    k1 = f( t(i-1), y(:,i-1) );
    k2 = f( t(i-1)+h/2, y(:,i-1)+(h/2)*k1 );
    k3 = f( t(i-1)+h/2, y(:,i-1)+(h/2)*k2 );
    k4 = f( t(i-1)+h, y(:,i-1)+h*k3 );

    y(:,i) = y(:,i-1) + h*(g(1)*k1+g(2)*k2+g(3)*k3+g(4)*k4);
end

end
```