SOLUTION 2 – Advanced Numerical Analysis

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1 ► Higher-order derivatives

a) $f(x+h) = \|x+h\|^2 = f(x) + 2\langle x,h\rangle + \|h\|^2$

$\Rightarrow f'(x)h = 2\langle x,h\rangle \quad (\nabla f(x) = 2x)$

$f''(x)[h,h] = 2\|h\|^2$

b) $f(A+H) = (A+H)^3 = A^3 + A^2H + AHA + HA^2 + AH^2 + HAH + H^2A + H^3$

$\Rightarrow f'(A)[H] = A^2H + AHA + HA^2$

$f''(A)[H,H] = 2(AH^2 + HAH + H^2A)$

c) $f(x+h) = \|x+h\|^{-2}(x+h) = (\|x\|^2 + 2\langle x,h\rangle + \|h\|^2)^{-1}(x+h)$

Hint

$$= \left(\|x\|^{-2} - \|x\|^{-4}(2\langle x,h\rangle + \|h\|^2) + \tfrac{1}{2}\|x\|^{-6}(2\langle x,h\rangle + \|h\|^2)^2 + \mathcal{O}(|h|^3)\right)(x+h)$$

$$= f(x) + \underbrace{\|x\|^{-2}(h - 2\|x\|^{-2}\langle x,h\rangle x)}_{\text{order } h} + \underbrace{\|x\|^{-4}(-2\langle x,h\rangle h - \|h\|^2x + 2\|x\|^{-2}|\langle x,h\rangle|^2 x)}_{\text{Order } h^2} + \mathcal{O}(|$$

$\Rightarrow f'(x)[h] = \|x\|^{-2}\left(h - 2\langle \frac{x}{\|x\|}, h\rangle \frac{x}{\|x\|}\right) = \|x\|^{-2}(I - 2P_x)h$, where P_x is the orthogonal projection on span $\{x\}$.

$f''(x)[h,h] = \|x\|^{-3}\left(4|\langle \frac{x}{\|x\|}, h\rangle|^2 \frac{x}{\|x\|} - 4\langle \frac{x}{\|x\|}, h\rangle h - 2\|h\|^2 \frac{x}{\|x\|}\right)$

2 ► Validating derivatives calculated by hand

```
function ex2problem2

f = @(A) inv(A);

% correct derivatives
df = @(A,H) -inv(A)*H*inv(A);
d2f = @(A,H) 2*inv(A)*H*inv(A)*H*inv(A);

% make the test for some random matrix 5x5 (almost surely invertible)
A = random('Normal',0,1,5,5);
close all;
check_derivatives(f,df,A,100,d2f);

% Test some wrong candidates
% Canditates could have been
dfwrong = @(A,H) -inv(A)^2*H;
d2fwrong = @(A,H) -2*inv(A)^3*H^2

% Scenario 1: df correct, d2f incorrect
figure
check_derivatives(f,df,A,100,d2fwrong);
```

```
% Scenario 2: df incorrect, d2f correct
figure
check_derivatives(f,dfwrong,A,100,d2f);

end

% Check derivatives for f : R^m to R^d.

function check_derivatives(f,df,x,N,d2f)

dim = size(x);

err1 = ones(1,N);
err1rand = ones(1,N);
err2 = ones(1,N);
err2rand = ones(1,N);
h = random('Normal',0,1,dim);   % choose a directional derivative

for i=1:N
    % fixed direction
    h = h/(norm(h)*i); %norm i^-1

    % varying hs
    hrand = random('Normal',0,1,dim); hrand = hrand/(norm(hrand)*i);

    %Taylor polynomial
    T = f(x+h) - f(x) - df(x,h);
    Trand = f(x + hrand) - f(x) -  df(x,hrand);

    % errors
    err1(i) = norm(T);
    err1rand(i) = norm(Trand);

    if nargin == 5  %if second derivative present
        err2(i) = norm(T - d2f(x,h)/2);
        err2rand(i) = norm(Trand - d2f(x,hrand)/2);
    end
end

% plots
% -----

%first order
subplot(1,2,1);

loglog(err1,'red'); hold on
loglog(err1rand);

%control curve for first order error should decay faster than N^(-2)
comp = [1:N].^(-2) * max(get(gca,'ylim')) * 10;
loglog(comp,'black');
grid on;

%same for second order
if nargin == 5
    subplot(1,2,2);

    loglog(err2,'red'); hold on
    loglog(err2rand,'blue');

    %control curve for first order error should decay faster than N^(-3)
    comp = [1:N].^(-3) * max(get(gca,'ylim')) * 10;
    loglog(comp,'black');
    grid on;
end
end
```