ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

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1 ► Constrained least squares problem

Let $\mathbf{y} \in \mathbb{R}^N$, $A \in \mathbb{R}^{N \times r}$, $r \leq N$, rank $A = r$. Consider the constrained minimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^N} \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|_2^2, \quad A^T\mathbf{x} = 0, \quad 1 - \|\mathbf{x}\|_2^2 \geq 0.$$

(a) Write down the KKT conditions.

(b) Assume $\|\mathbf{y}\|_2 < 1$. Show that the inequality constraint is inactive at a global solution $\mathbf{x}^*$ and deduce that

$$\mathbf{x}^* = (I - A(A^TA)^{-1}A^T)\mathbf{y}.$$

$\min \frac{1}{2}\|x-y\|^2$, $A^Tx = 0$, $1 - \|x\|^2 \geq 0$

$\to$ $\nabla f(x) = x - y$, $\nabla h(x) = A$, $\nabla g(x) = -x$

a) KKT conditions

(1) $x^* - y = A\mu - \lambda x^*$, $\mu \in \mathbb{R}^r$, $\lambda \in \mathbb{R}$

(2) $A^Tx^* = 0$

(3a) $\|x^*\| \leq 1$, (3b) $\lambda \geq 0$, (3c) $\lambda(1 - \|x^*\|^2) = 0$

b) For fixed $x \in \ker A^T$, $\alpha \mapsto \|\alpha x - y\|^2 = \alpha^2\|x\|^2 - 2\alpha\langle x, y\rangle + \|y\|^2$ is minimal for $\alpha^* = \frac{\langle x, y\rangle}{\|x\|^2}$. Hence $\|\alpha^* x\| \leq \|y\|$.

We conclude that any global minimizer of our problem will satisfy $\|x\| \leq \|y\|$, i.e., if $\|y\| < 1$ the inequality constraint is inactive. ($\lambda = 0$ by (3c))

$\to$ Reduced KKT: (1') $x^* - y = A\mu$, (2) $A^Tx^* = 0$.

Since A is of full column rank, A^TA is invertible. Thus (1') gives

$$\mu = (A^TA)^{-1}\underbrace{A^T(x^*}_{=0} - y) = -(A^TA)^{-1}A^Ty.$$

Inserting in (1') again yields

$$\underline{\underline{x^* = (I - A(A^TA)^{-1}A^T)y}}$$

Why is this the solution?

1. The problem has a solution (Ω is compact)
2. LICQ holds since $\nabla h(x) = A$ has full rank and g is inactive.

$\overset{2.}{\Rightarrow}$ 3. KKT is necessary! Since there is only one solution to KKT, it is the global solution to the problem

2 ▶ Constrained optimization

*Note: This is an old exam question (without part **(a)**).*

Consider the set

$$\Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid g(\mathbf{x}) \geq 0\} \quad \text{where} \quad g(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_2 \\ 1 - x_1 - x_2 \end{pmatrix}.$$

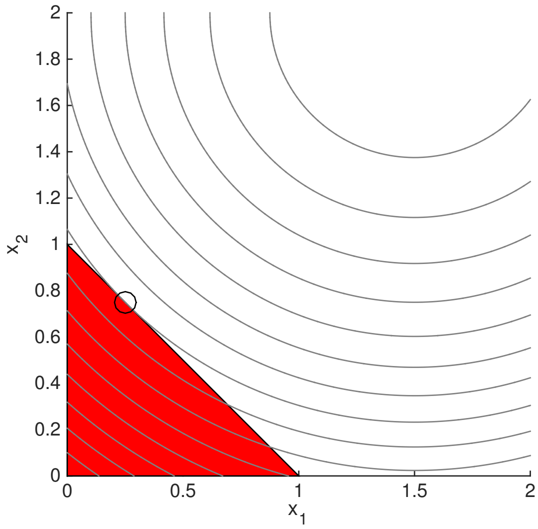
Solve the constrained optimization problem

$$\min_{\mathbf{x} \in \Omega} f(\mathbf{x}) \quad \text{where} \quad f(\mathbf{x}) = (x_1 - 3/2)^2 + (x_2 - 2)^2 \tag{1}$$

by following these steps:

(a) Plot the constraint set defined by $g(\mathbf{x})$ in the (x_1, x_2) plane and add contour lines of the cost function. Can you already guess the optimal solution?

```
patch([0,0,1,0],[0,1,0,0],'r')
hold on
contour(X,Y,f(X,Y),15,'color',[0.5,0.5,0.5]);
plot(0.25,0.75,'ok','markersize',16)
axis equal
xlabel('x_1')
ylabel('x_2')
set(gca,'fontsize',14)
```



From the plot, we guess that the optimal solution is approximately at $(0.25, 0.75)$.

(b) Write down the KKT conditions for problem (1).

We have

$$\nabla f(\mathbf{x}) = \begin{pmatrix} 2(x_1 - 3/2) \\ 2(x - 2) \end{pmatrix}, \quad \nabla g_1(\mathbf{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \nabla g_2(\mathbf{x}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \nabla g_3(\mathbf{x}) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Therefore the main KKT equation $\nabla f(\mathbf{x}) = \sum_i \nabla \lambda_i g_i(\mathbf{x})$ *reads*

$$\begin{pmatrix} 2(x_1 - 3/2) \\ 2(x_2 - 2) \end{pmatrix} = \begin{pmatrix} \lambda_1 - \lambda_3 \\ \lambda_2 - \lambda_3 \end{pmatrix}.$$

Additionally, one requires

$$g(\mathbf{x}) \geq 0, \quad \lambda \geq 0, \quad \lambda_i g_i(\mathbf{x}) = 0 \quad \textit{for } i = 1, 2, 3.$$

(c) Show that LICQ holds for all $\mathbf{x} \in \Omega$.

There is no point in Ω where all three constraints are active. Since any two vectors out of $\nabla g_1(\mathbf{x})$, $\nabla g_2(\mathbf{x})$, $\nabla g_3(\mathbf{x})$ are linearly independent, LICQ always holds.

(d) Find all KKT points for which at most one constraint is active.

There is no point in Ω where all three constraints are active. Since any two vectors out of $\nabla g_1(\mathbf{x})$, $\nabla g_2(\mathbf{x})$, $\nabla g_3(\mathbf{x})$ are linearly independent, LICQ always holds.

If we assume that at most one constraint is active, then by the complimentary condition at least two Lagrange multipliers have to vanish. However, if $\lambda_1 = \lambda_3 = 0$ or $\lambda_2 = \lambda_3 = 0$ we deduce from the first KKT equation $x_1 = 3/2$ or $x_2 = 2$, respectively, which both would violate $g(\mathbf{x}) \geq 0$. Thus, the only option is $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 \neq 0$. This gives

$$x_1 - 3/2 = -\lambda_3/2 = x_2 - 2.$$

Since g_3 is active, we have the additional equation

$$x_1 + x_2 = 1.$$

The unique solution of both equations is $\mathbf{x}^* = \begin{pmatrix} 1/4 \\ 3/4 \end{pmatrix}$.

(e) Find all $\mathbf{x} \in \Omega$ at which at least two constraints are active.

Only at three points two constraints are active: $\mathbf{x}_1 = 0$ with $f(\mathbf{x}_1) = 25/4$, $\mathbf{x}_2 = (1, 0)^T$ with $f(\mathbf{x}_2) = 17/4$, and $\mathbf{x}_3 = (0, 1)^T$ with $f(\mathbf{x}_3) = 13/4$.

(f) Select the solution of the problem from the points in (d) and (e).

Since all function values in (e) are larger than $f(\mathbf{x}^*) = 25/8$, since $\mathbf{x}^*$ is the unique KKT point with at most one active constraint, and since the problem must have a solution, the solution is $\mathbf{x}^*$