SOLUTION 10 – Advanced Numerical Analysis

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

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1 ▶ Tangent cone and linearized feasible directions

Consider again the tangent cone $T_\Omega(\mathbf{x})$ from the last exercise sheet:

$$\mathbf{x} = (-1, 0, 0)^T, \quad \Omega = \{\mathbf{x} \in \mathbb{R}^3 \colon x_1^2 + x_2^2 \le 1,\ x_3^2 = 0\}.$$

Sketch $\mathcal{F}(\mathbf{x})$. Modify the parametrization of Ω such that $T_\Omega(\mathbf{x}) = \mathcal{F}(\mathbf{x})$ holds.

Follows directly from the definition.

2 ▶ Lagrange multipliers

Show that the Lagrange multipliers λ^* and μ^* in Theorem 5.11 are unique.

If LICQ holds, then $[g'(x^*)|_{\mathcal{A}}, h'(x^*)]^T$ has full row rank.
The existence of a solution of the linear system

$$[g'(x^*)|_{\mathcal{A}}, h'(x^*)] \begin{bmatrix} \lambda^* \\ \mu^* \end{bmatrix} = \nabla f(x^*)$$

is hence equivalent to the existence of a unique solution.

3 ▶ KKT conditions

Find the point(s) on the parabola $y = \frac{1}{5}(x-1)^2$ which is/are closest to $(x, y) = (1, 2)$ in the Euclidian norm, i.e., solve

$$\min_{x,y} (x-1)^2 + (y-2)^2 \quad \text{subject to} \quad (x-1)^2 = 5y.$$

Find all KKT points, and the solution. Check that replacing $(x-1)^2$ in the objective function by $5y$ and solving the unconstrained problem for y does not work.

a) The first KKT condition is

$$\begin{pmatrix} 2(x_1 - 1) \\ 2(x_2 - 2) \end{pmatrix} = \mu^* \begin{pmatrix} 2(x_1 - 1) \\ -5 \end{pmatrix}$$

Either $x_1 = 1$, then $x_2 = 0$ (by constraint) $\to f(x_1^*, x_2^*) = 4$,
Or $\mu^* = 1$, but then $x_2 = -\frac{1}{2}$ which is not possible (by constraint

$\Rightarrow$ only one KKT point $(1, 0)$.

b) If we substitute $(x-1)^2 = 5y$ into f we get

$$\tilde{f}(y) = 5y + (y-2)^2$$

Minimizing this function does not work, since

$$\tilde{f}'(y) = 5 - 2(y-2) \stackrel{!}{=} 0$$

only gives $y = -\frac{1}{2}$, which does not solve the problem.

4 ▶ KKT conditions

For a fixed parameter t consider

$$\min_{\mathbf{x}\in\mathbb{R}^2} \left(x_1 - \frac{3}{4}\right)^2 + (x_2 - t)^4 \quad \text{subject to} \quad \begin{bmatrix} 1 - x_1 - x_2 \\ 1 - x_1 + x_2 \\ 1 + x_1 - x_2 \\ 1 + x_1 + x_2 \end{bmatrix} \geq 0.$$

(a) For what values of t, if there are any, does $\mathbf{x} = (1, 0)^T$ satisfy the KKT conditions?

(b) Show that when $t = 1$, only the first constraint is active at the solution, and find the solution.

$$\min \ (x_1 - \tfrac{3}{4})^2 + (x_2 - t)^4 \qquad \text{s.t.} \quad \begin{pmatrix} 1 - x_1 - x_2 \\ 1 - x_1 + x_2 \\ 1 + x_1 - x_2 \\ 1 + x_1 + x_2 \end{pmatrix} \geq 0$$

a) $x = (1,0)^T$: only the first two constraints active

$$\nabla f(x) = \begin{pmatrix} 2(x_1 - \frac{3}{4}) \\ 4(x_2 - t)^3 \end{pmatrix}, \quad \nabla g_1(x) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad \nabla g_2(x) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\rightarrow$ KKT:
$$\begin{pmatrix} 2(\cancel{x_1}1 - \frac{3}{4}) \\ 4(0 - t)^3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -4t^3 \end{pmatrix} = \begin{pmatrix} -\lambda_1 - \lambda_2 \\ -\lambda_1 + \lambda_2 \end{pmatrix}$$

The first row violates $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ (3b)).
Hence for no value of t $x = (1,0)$ can be a KKT point.

b) $t = 1$: $\min \ (x_1 - \frac{3}{4})^2 + (x_2 - 1)^4$

It is obvious that the minimum is attained for some $x_1 \geq 0, x_2 \geq 0$

$f(1,0) = \frac{1}{16} + 1$, $f(0,1) = \frac{9}{16}$, $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{16} + \frac{1}{16}$

$\rightarrow$ the minimum is not attained for $(0,1)$ or $(1,0)$
$\rightarrow$ really only first constraint active and $x_1 > 0$, $x_2 > 0$

$\rightarrow$ KKT:
$$\begin{pmatrix} 2(x_1 - \frac{3}{4}) \\ 4(x_2 - 1)^3 \end{pmatrix} = \lambda_1 \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$\lambda_1 = 0$ is not possible, since $(\frac{3}{4}, 1)$ does not satisfy the constr.

$\rightarrow \quad 2(x_1 - \frac{3}{4}) = 4(x_2 - 1)^3$, $x_1 + x_2 = 1$

$\xrightarrow{x_2 - 1 = -x_1} \quad 4x_1^3 + 2x_1 - \frac{3}{2} = 0 \quad \Rightarrow \quad x_1 = \frac{1}{2} \quad \Rightarrow \quad x_2 = \frac{1}{2} \quad \Rightarrow \quad x^* = (\frac{1}{2}, \frac{1}{2})$

5 ► Geometric and arithmetic mean

(a) Use the KKT conditions to find the global solution of

$$\min_{\mathbf{x}\in\mathbb{R}^n} \sum_{i=1}^n x_i \quad \text{subject to} \quad \prod_{i=1}^n x_i = 1,\ \mathbf{x} \geq 0.$$

Hint. First show that the global minimizer satisfies $\mathbf{x} > 0$.

(b) Use (a) to prove the inequality between the geometric and the arithmetic mean:

$$\left(\prod_{i=1}^n x_i\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^n x_i \quad \text{for all } \mathbf{x}\in\mathbb{R}^n \text{ with } \mathbf{x} \geq 0.$$

a) $\min\limits_{x\in\mathbb{R}^n} \sum\limits_{i=1}^n x_i$ s.t. $\prod\limits_{i=1}^n x_i = 1$, $x \geq 0$

At the solution obviously $x > 0$, since otherwise constraints cannot be satisfied ($\prod x_i = 0$ if one $x_i = 0$)

→ KKT involves only $h(x) = \prod x_i - 1$:

$$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \nabla f(x) = \mu \nabla h(x) = \mu \begin{pmatrix} x_2 \cdots x_n \\ x_1 x_3 \cdots x_n \\ \vdots \\ x_1 \cdots x_{n-1} \end{pmatrix}$$

→ $\mu \neq 0$ and since $\prod x_i = 1$, every row implies

$$|x_i| = \frac{1}{\prod_{j\neq i} x_j} = \frac{1}{\mu} \Rightarrow \mu = 1, \quad x_i = 1 \quad \forall i$$

→ minimum is attained at $x = (1, .., 1)$ and takes the value

$$f(1, .., 1) = n.$$

b) For any $x > 0$ we have that y with

$$y_i = x_i \left(\prod_{j=1}^n x_j\right)^{-1/n}$$

satisfies $\prod y_i = 1$. Hence by a)

$$\sum_{i=1}^n y_i \geq n,$$

or, rearranging,

$$\left(\prod_{i=1}^n x_i\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^n x_i.$$

If one $x_i = 0$, this inequality trivially holds. ∎