ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

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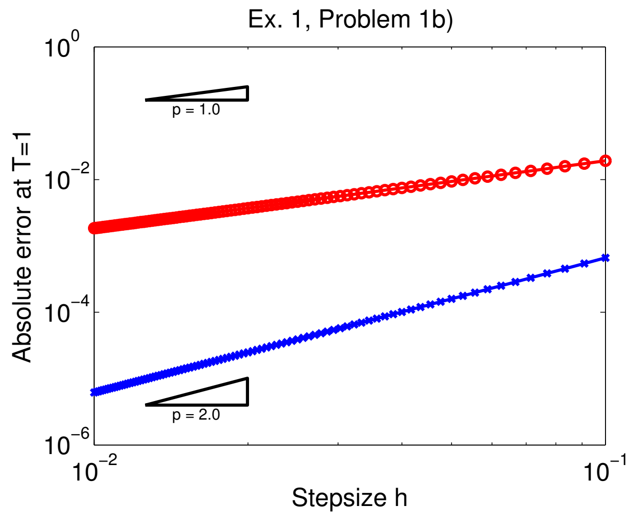
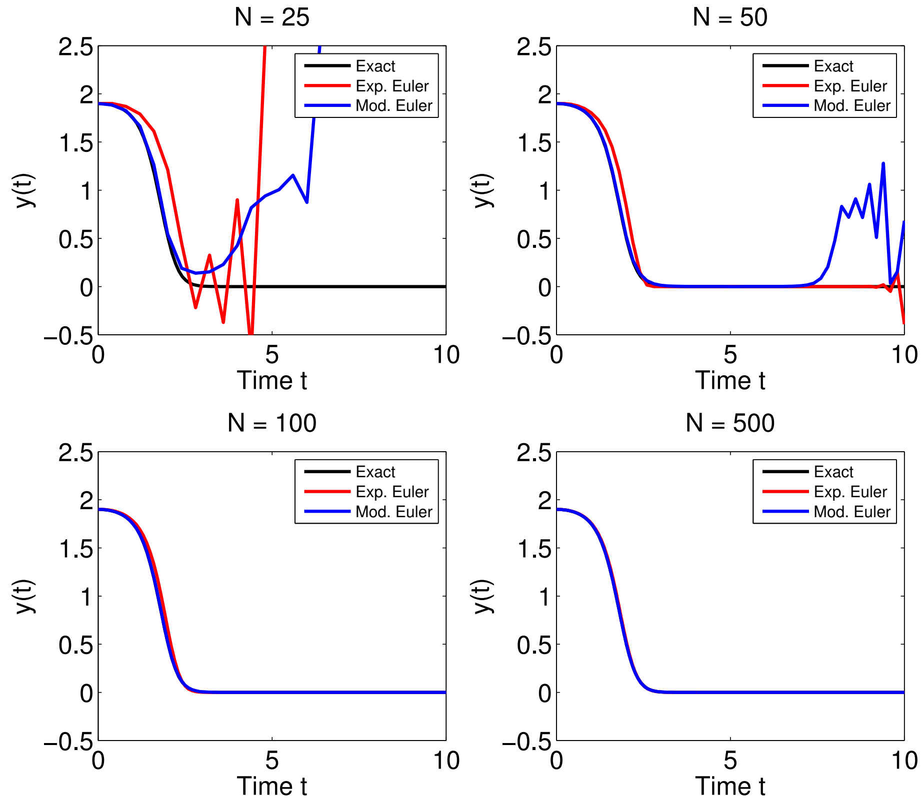
1 ▶ Euler and modified Euler method

```
function ex1problem1

close all
set(0,'defaultLineLineWidth', 2)
set(0,'DefaultAxesFontSize',16)

% Part (b)
% --------

N = [10:1:100];
f = @(t,y) -y;
y0 = 1;
I = [0,1];
for i=1:size(N,2)
    [T1,Y1] = expeuler(f,y0,I,N(i));
    [T2,Y2] = modeuler(f,y0,I,N(i));
    Exact = exp(-T1);
    Error1(i) = abs(Y1(end) - Exact(end));
    Error2(i) = abs(Y2(end) - Exact(end));
end
figure(1);
set(gca,'fontsize',16)
loglog(1./N, Error1, '-or')
hold on
loglog(1./N, Error2, '-xb')
hold off
xlabel('Stepsize h')
ylabel('Absolute error at T=1')
title('Ex. 1, Problem 1b)')
% determine convergence rates and include them in plot
p1 = polyfit(log(1./N), log(Error1), 1)
p2 = polyfit(log(1./N), log(Error2), 1)

% Part (c)
% --------

f = @(t,y) t*y*(y-2);
y0 = 1.9;
exact = @(t) 2 ./ (1 - (y0-2)/y0 .* exp(t.^2) );
I = [0,10];
Iexact = linspace(I(1),I(2));
N = [25 50 100 500];

figure(2)
for i = 1:4
    subplot(2,2,i);
    [T1,Y1] = expeuler(f,y0,I,N(i));
    [T2,Y2] = modeuler(f,y0,I,N(i));
    plot(Iexact,exact(Iexact),'-k');
    hold on
    plot(T1,Y1,'-r');
    plot(T2,Y2,'-b');
    hold off
    title(['N = ' num2str(N(i)) ])
    xlabel('Time t')
    ylabel('y(t)')
    l = legend('Exact','Exp. Euler','Mod. Euler');
    set(l,'fontsize',10)
    ylim([-0.5,2.5])
end

```

```
end

% Part (a)
% --------

function [T,Y] = expeuler(f, y0,I, N)
    Y(1) = y0; h = (I(2)-I(1))/N; T = I(1)*ones(1,N+1) + h*(0:N);
    for i =1:N
        Y(i+1) = Y(i) + h*f(T(i),Y(i));
    end
end

function [T,Y] = modeuler(f, y0,I, N)
    Y(1) = y0; h = (I(2)-I(1))/N; T = I(1)*ones(1,N+1) + h*(0:N);
    for i =1:N
        eta = Y(i) + (h/2)*f(T(i),Y(i));
        Y(i+1) = Y(i) + h*f(T(i)+h/2,eta);
    end
end
```





Exercise 1

1d) The exact solution of

$$\begin{cases} y'(t) = t\,y(t)\,(y(t)-2) \\ y(0) = y_0 \end{cases}, \quad t \in [0,10]$$

can be obtained by separation of variables:

$$\int_0^t \frac{y'(s)}{y(s)(y(s)-2)}\,ds = \int_0^t s\,ds$$

$$\rightsquigarrow \int_{y_0}^{y(t)} \frac{1}{s(s-2)}\,ds = \frac{1}{2}t^2 \quad , \quad \frac{1}{s(s-2)} = \frac{1}{2}\left(\frac{1}{s-2} - \frac{1}{s}\right)$$

$$\rightsquigarrow \frac{1}{2}\Big[\ln|s-2| - \ln|s|\Big]_{y_0}^{y(t)} = \frac{1}{2}t^2$$

$$\rightsquigarrow \ln\left|\frac{y(t)-2}{y(t)}\right| - \ln\left|\frac{y_0-2}{y_0}\right| = t^2$$

$$\rightsquigarrow 1 - \frac{2}{y(t)} = \frac{y(t)-2}{y(t)} = \left(\frac{y_0-2}{y_0}\right)e^{t^2}$$

$$\rightsquigarrow y(t) = \frac{2}{1 - \left(\frac{y_0-2}{y_0}\right)e^{t^2}}$$

As you can see, these derivations become problematic when $y(t) = 2$ for some t. Actually, the only possibility for this is $y_0 = 2$.

2 ► Charged particle in electromagnetic field

```
function ex1problem2
    [T,Y] = ode45(@magnetic,[0 10],[1 1 1 1 1 1]);
    comet3(Y(:,1),Y(:,2),Y(:,3),.1);
end


% Define  the right-hand side
function dy = magnetic(t,y)
    mu = [0; 0; 100];
    dy = zeros(6,1);

    x = y(1:3); dotx = y(4:6);
    r = norm(x,2);
    B = (3 * (x' * mu)*x - mu*r^2) / r^5;

    dy(1:3) = dotx;
    dy(4:6) = cross(B,dotx);
end
```

3 ► Gronwall's lemma

3a) Claim: $(u_n),(w_n) \geq 0$: $u_n \leq \alpha + \sum_{k=0}^{n-1} u_k w_k \Rightarrow u_n \leq \alpha \exp\left(\sum_{k=0}^{n-1} (1+w_k)\right)$

(ii) Prove sharper inequality: $u_n \leq \alpha \prod_{k=0}^{n-1} (1+w_k)$

Proof by induction ($n=0$ clear) is immediate from (i). ☐

(iii) The claim follows from (ii) and $(1+x) \leq \exp(x)$ for $x \geq 0$.

b) Claim: $u,w \geq 0$ continuous, $u(t) \leq \alpha + \int_{t_0}^{t} u(s)w(s)ds \Rightarrow u(t) \leq \alpha \exp\left(\int_{t_0}^{T} w(s)ds\right)$

Proof: Let $\varepsilon > 0$. There ex. $N > 0$ s.t. $|t-s| < \frac{1}{N}$ implies $|u(t) - u(s)| < \varepsilon$.

Define $I_n = [t_0 + n\frac{(T-t_0)}{N}, t_0 + (n+1)\frac{(T-t_0)}{N}]$, $n = 0, \ldots, N-1$ and

$u_n = \max_{t \in I_n} u(t)$, $w_n = \max_{t \in I_n} w(t)$. Then it follows from the assumption that

$$u(t) \leq \alpha + \frac{1}{N} \sum_{k=0}^{n-1} u_k w_k \quad \text{for all } t \leq t_0 + \frac{n}{N}.$$

In particular,

$$u_n \leq \alpha + \frac{1}{N} \sum_{k=0}^{n-1} u_k w_k.$$

So by (a),

$$u_n \leq \alpha \exp\left(\frac{1}{N} \sum_{k=0}^{N-1} u_k w_k\right).$$

Now let $t \in I_n$, then

$$u(t) \leq u_n + |u(t) - u_n| \leq \alpha \exp\Big(\underbrace{\frac{1}{N} \sum_{k=0}^{N-1} u_k w_k}_{\to \int_{t_0}^{t} u(s)w(s)ds \text{ for } N \to \infty}\Big) + \varepsilon$$ ☐

c)

$$|y(t) - \tilde{y}(t)| \leq |y_0 - \tilde{y}_0| + \int_a^t L|y(s) - \tilde{y}(s)|ds + (b-a)\|\sigma\|_\infty$$

$$\overset{b)}{\Rightarrow} |y(t) - \tilde{y}(t)| \leq \left(|y_0 - \tilde{y}_0| + (b-a)\|\sigma\|_\infty\right) \cdot \exp(L(b-a)).$$ ☐