ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

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1 ► Quiz

Give the correct answer.

(a) Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then

$$\text{(i)}\quad f'(\mathbf{y}) = 0, \quad \text{(ii)}\quad f'(\mathbf{y}) = f, \quad \text{(iii)}\quad f'(\mathbf{y}) = f(\mathbf{y}).$$

(b) Let $f(\mathbf{y}) = \frac{1}{2}\mathbf{y}^T A\mathbf{y}$. Then $f''(\mathbf{y})[\mathbf{h},\mathbf{k}] = \mathbf{h}^T A\mathbf{k}$. (i) True (ii) False

(c) For the function $f(y) = \cos y$ and the rooted tree $\beta = [\odot, [\odot, \odot], [[\odot]]]$,

$$\text{(i)}\quad f^{(\beta)}(0) = 0, \quad \text{(ii)}\quad f^{(\beta)}(0) = 1.$$

(d) For $f(\mathbf{y}) = \frac{1}{2}\mathbf{y}^T A\mathbf{y}$ and the rooted tree $\beta = [\odot, \odot, [\odot]]$

$$\text{(i)}\quad f^{(\beta)}(\mathbf{y}) = A^3\mathbf{y}, \quad \text{(ii)}\quad f^{(\beta)}(\mathbf{y}) = \mathbf{y}, \quad \text{(iii)}\quad f^{(\beta)}(\mathbf{y}) = 0.$$

(e) The Runge-Kutta method

$$\begin{array}{c|cc} 1 & 1 & \\ 3/4 & 1/4 & 1/2 \\ \hline & 1/2 & 1/4 \end{array}$$

is consistent.

(i) True (ii) False

(f) Let G be symmetric and invertible. Then the IVP $\dot{\mathbf{y}}(t) = G\mathbf{y}(t)$, $\mathbf{y}(t_0) = \mathbf{y}_0$ is stable if and only it is asymptotically stable.

(i) True (ii) False (iii) One cannot say.

2 ► Implementing steepest descent

(a) Write a MATLAB function `x = steepdesc(f,df,x0,c1,tol)` that implements the steepest descent algorithm

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \alpha_k \nabla f(\mathbf{x}_k).$$

Use backtracking with $\beta = 1/2$ for selecting a step size α_k which satisifes the Armijo condition

$$f(\mathbf{x}_k - \alpha_k \nabla f(\mathbf{x}_k)) \le f(\mathbf{x}_k) - c_1 \alpha_k \|\nabla f(\mathbf{x}_k)\|_2^2.$$

The output x contains the iterates as columns. The iteration stops if $\|\nabla f(x_k)\|_2 \le$ `tol`.

(b) Test your program with the function $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - b^T\mathbf{x}$ where

$$A = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Use, e.g., `c1 = 1e-4`, `tol = 1e-6`, and `x0 = [3; 4.5]`. Use the code lines provided on the homepage to visualize what happened. Compare to backtracking with $\beta = 0.1$.

(c) Now repeat the task, but replace the step length by the choice

$$\alpha = \frac{\|A\mathbf{x} - b\|_2^2}{(A\mathbf{x} - b)^T A(A\mathbf{x} - b)} = \frac{\|\nabla f(\mathbf{x})\|_2^2}{\nabla f(\mathbf{x})^T A \nabla f(\mathbf{x})}.$$

Prove that this choice is optimal for quadratic problems $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - b^T\mathbf{x}$ with symmetric positive definite A.

(d) Repeat the task by always using step size $\alpha = 0.1$.

3 ▶ Descent directions are not enough

For $f\colon \mathbb{R}^2 \to \mathbb{R},\ x \mapsto \frac{1}{2}\|\mathbf{x}\|^2$, consider the line search method with directions

$$\mathbf{p}_k = \mathbf{q}_k - \frac{1}{2^{k+3}}\nabla f(\mathbf{x}_k),$$

where $\mathbf{q}_k$ is some vector orthogonal to $\nabla f(\mathbf{x}_k)$ such that $\|\mathbf{p}_k\| = \|\nabla f(\mathbf{x}_k)\|$.

Show that the $\mathbf{p}_k$ are descent directions, but that for every starting point $\mathbf{x}_0$ and every nonnegative sequence of step-sizes α_k the critical point $\mathbf{x}^* = 0$ is not a cluster point of the sequence $(\mathbf{x}_k)$.