1 Order of implicit methods

For autonomous IVPs, find and prove the consistency order of the implicit midpoint rule given by the table
\[
\begin{array}{c|c|c}
1/2 & 1/2 & 1 \\
\end{array}
\]

2 Implementation of implicit RK methods

(a) Write a function
\[
z = \text{newton}(F,dF,z0,atol,maxiter)
\]
which implements Newton’s method
\[
DF(z^k)\Delta z^k = -F(z^k), \quad z^{k+1} = z^k + \Delta z^k,
\]
starting from \(z^0\), and terminating after \(maxiter\) steps, or when the criterion
\[
\frac{\theta^k}{1 - \theta^k} \|\Delta z^k\| \leq \text{atol}, \quad \theta^k = \frac{\|\Delta z^k\|}{\|\Delta z^{k-1}\|}
\]
is satisfied (set \(\theta_1 = 1\)). If you find this boring you might download the method from the homepage.

(b) Implement the implicit trapezoidal rule as a Matlab function. It could have the syntax
\[
[t,y] = \text{itrapez}(f,tspan,y0,N,fjac,atol,maxiter)
\]
where \(fjac(t,y)\) should be the Jacobian of \(y \mapsto f(t,y)\).

For a general implicit RK method at iterate \(y_n\), let \(G: \mathbb{R}^{ds} \to \mathbb{R}^{ds}\) be given by
\[
G(z) = \begin{pmatrix}
\sum_{j=1}^s a_{1j} f(t + c_j h, y_n + z_j) \\
\vdots \\
\sum_{j=1}^s a_{sj} f(t + c_j h, y_n + z_j)
\end{pmatrix} \in \mathbb{R}^{ds}, \quad z = \begin{pmatrix}
z_1 \\
\vdots \\
z_s
\end{pmatrix} \in \mathbb{R}^{ds}.
\]
The Newton method from (a) has to be called to solve for \(F(z) = z - hG(y_n,z) = 0\) using \(z^0 = 0\) as a starting value.\(^{1}\) The next step then is
\[
y_{n+1} = y_n + h \sum_{j=1}^s b_j f(t + c_j h, y_n + z_j).
\]

(c) Consider an interaction of three species modelled by
\[
y' = f(t,y) = \begin{pmatrix}
-0.04y_1 + 10^4 y_2 y_3 \\
0.04y_1 - 10^4 y_2 y_3 - 3 \cdot 10^7 y_2^2 \\
3 \cdot 10^7 y_2^2
\end{pmatrix}, \quad y(0) = \begin{pmatrix}1 \\ 0 \\ 0 \end{pmatrix}.
\]

(i) Plot the solution component \(y_2\) you obtain with \(\text{itrapez}\) on the time interval \([0,0.3]\) using \(N = 20, 50, 100, 200\) for \(\text{atol} = 10^{-4}\) and \(10^{-6}\) (use \(\text{maxiter} = 10\)). Compare to the explicit Runge-Kutta method for these values of \(N\).

(ii) Now call the explicit \texttt{ode45}\(^{1}\) and the implicit \texttt{ode23s} from Matlab, which both have a step size control. Try
\[
\text{ode45}(f,[0,0.3],[1 0 0],\oderset('Stats','on')), \\
\text{ode23s}(f,[0,0.3],[1 0 0],\oderset('Stats','on')), \text{and} \\
\text{ode23s}(f,[0,0.3],[1 0 0],\oderset('Stats','on','Jacobian',\text{fjac}))
\]
(where \(\text{fjac}\) should be the function handle of the Jacobian). Have a look on the number of steps sizes and number of rejected steps.

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\(^{1}\)In case of the implicit trapezoidal rule, if \(t_n\) and \(y_n\) are the current values, this amounts in, e.g., passing the following two function handles to the Newton method:
\[
F = @(z) z - h*\text{zeros}(d,1); f(tn+1,y + z(1:d))/2 + f(tn + h,yn + z(d+1:2*d))/2;
\]
\[
dF = @(z) \text{eye}(2*d) - h*[\text{zeros}(d,2*d); \text{fjac}(tn+1,y + z(1:d))/2, \text{fjac}(tn + h,yn + z(d+1:2*d))/2];
\]
3 Stability of rational approximation

Assume the linear ODE

\[ y' = Ay \]

is stable. For a rational approximation \( S \) of the exponential function of consistency order \( p \) show the following: If the characteristic step size \( h_c \) is positive, then there exists \( C > 0 \) such that

\[ \| \exp(nhA) - S(hA)^n \| \leq Ch^p \quad \text{for all } 0 < h < h_c. \]