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## 1 ► Order criteria

Consider an explicit two-stage Runge-Kutta method given by the Butcher table

$$\begin{array}{c|cc} 0 & 0 & 0 \\ c & a & 0 \\ \hline & b_1 & b_2 \end{array}$$

Write down equations for the parameters such that the method is consistent of order two. Show that the order of consistency cannot be higher.

## 2 ► Simpler order conditions for simple ODEs

Let $(A, \mathbf{b}, \mathbf{c})$ be an explicit $s$-stage Runge Kutta method, $s \geq p$, which is invariant under autonomization, i.e., $\mathbf{c} = A\mathbf{e}$.

(a) Consider the simplest autonomous initial value problem

$$\mathbf{y}'(t) = B\mathbf{y}(t), \quad \mathbf{y}(t_0) = \mathbf{y}_0 \in \mathbb{R}^d, \quad B \in \mathbb{R}^{d\times d}.$$

Show that the Runge-Kutta method has consistency order $p$ if and only if

$$\mathbf{b}^\mathsf{T} A^{(\beta)} = \frac{1}{(\#\beta)!} \quad \text{for all linear trees } \beta = [[\cdots[\odot]\cdots]] \text{ with } \#\beta \leq p.$$

(b) Consider the quadrature problem

$$\mathbf{y}'(t) = f(t), \quad \mathbf{y}(t_0) = \mathbf{y}_0 \in \mathbb{R}^d$$

of a function $f \in C^\infty([0, T])$. Show that the Runge-Kutta method has consistency order $p$ if and only if

$$\mathbf{b}^\mathsf{T} A^{(\beta)} = \frac{1}{\#\beta} \quad \text{for all one-level trees } \beta = [\odot, \ldots, \odot] \text{ with } \#\beta \leq p.$$

*Hint.* What is $A^{(\beta)}$ for such trees?

## 3 ► Local vs. global error

In this exercise, we will investigate the difference between the local and the global error. We consider the ODE

$$y'(t) = 1 - t + 3y, \quad y(t_0) = y_0$$

with exact solution

$$y(t) = C\mathrm{e}^{3t} - \frac{2}{9} + \frac{t}{3}, \quad C(t_0, y_0) = \frac{y_0 - \frac{t_0}{3} + \frac{2}{9}}{\mathrm{e}^{3t_0}}.$$

1. Create a plot similiar to Figure 2.1 *(Lady Windemere's fan)* using an explicit Euler method on the interval $T = [0, 1]$ with stepsize $h = 0.2$ and initial condition $y(0) = 1$. To do this, you have to calculate the exact solutions $y(t)$ with initial conditions $y(t_i) = y_i$ obtained from the $i$th step of the explicit Euler scheme.
2. Plot the global error $|y(1) - y_N|$, where $y_N$ is the last step of the Euler scheme and compare it to the maximum local error, both as functions of the stepsize $h$. What is the convergence order that you obtain for the local and the global error?

## 4 ► Order isn't everything[1]

Solve the initial value problem

$$y'(t) = |1.1 - y| + 1, \quad t \in [0, 0.1], \quad y(0) = 1,$$

both analytically, and numerically using the explicit Euler method, Runge's method, and the classical RK4 method. Plot errors of all three methods against time.

Then write a script which does the same for various step sizes $N = 2^k$, $k = 1, 2, \ldots, 10$, and measures the runtime of each method (use the commands `tic` and `toc`). For all three methods, plot the runtime versus the achieved final accuracy at $T = 0.1$ (use a double logarithmic scale). You may compare to a plot of the required number of function evaluations versus the final achieved accuracy.

*Hint.* The exact solution is

$$y(t) = \begin{cases} -1.1e^{-t} + 2.1, & 0 \le t \le \ln 1.1, \\ \frac{10}{11}e^{t} + 0.1, & \ln 1.1 \le t \le 0.1. \end{cases}$$

[1] Title and exercise taken from Deuflhard, Bornemann: Scientific Computing with Ordinary Differential Equations, Springer, 2002.