ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

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1 ▶ First Runge-Kutta methods

Let $\lambda \in \mathbb{C}$. Consider the differential equation

$$y'(t) = \lambda y(t), \quad y(0) = y_0.$$

(a) Calculate the result x_1 of one step of Runge's method with stepsize h.

(b) Show that every s-stage explicit Runge-Kutta method of order s will produce

$$y_1 = \Big(\sum_{j=0}^{s} \frac{z^j}{j!}\Big) y_0, \quad z = h\lambda.$$

Hint. Obviously, for the above problem y_1 will be a polynomial in h of degree s.

2 ▶ Rooted trees

Draw the following rooted trees, calculate the tree factorial, and write out the represented differential:

$$[\odot, [\odot], [\odot, \odot]], \quad [[[[\odot], \odot], \odot, [\odot, \odot]]].$$

3 ▶ Verifying order of Runge-Kutta methods

(a) Given an explicit three-stage Runge Kutta method

$$\begin{array}{c|ccc} c_1 & 0 & 0 & 0 \\ c_2 & a_{21} & 0 & 0 \\ c_3 & a_{31} & a_{22} & 0 \\ \hline & b_1 & b_2 & b_3 \end{array},$$

write down the conditions such that it is invariant under autonomization and of order three. *Hint.* Use Table 2.1.

(b) Using the conditions found in (a), try to correct the following table changing as few entries as possible, such that the method is invariant under autonomization and of order three:

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & -\frac{1}{2} & \frac{3}{2} & 0 \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}.$$

4 ▶ A problem from astronomy

(a) Write a function `[t,y] = rk4(fun,tspan,y0,N)` that implements the classical Runge-Kutta method. Input parameters are a function handle, a time interval, the initial value and the number of steps.

(b) The following example is taken from: Hairer, Nørsett, Wanner: Solving Ordinary Differential Equations I, Springer 1993. It is an instance of the restricted three body problem, and one may think about a small particle flying in the plane under the influence of the earth clamped at the origin and the moon surrounding it on the unit circle. The plane has to be imagined as a relative coordinate system of the earth which rotates slowly around the sun. The equations of motion for the position $\mathbf{x} = (x_1, x_2)^T$ of the particle are

$$\ddot{\mathbf{x}} = \begin{pmatrix} x_1 + 2\dot{x}_2 \\ x_2 - 2\dot{x}_1 \end{pmatrix} - \frac{\tilde{\mu}}{((x_1+\mu)^2 + x_2^2)^{3/2}} \begin{pmatrix} x_1 + \mu \\ x_2 \end{pmatrix} - \frac{\mu}{((x_1-\tilde{\mu})^2 + x_2^2)^{3/2}} \begin{pmatrix} x_1 - \tilde{\mu} \\ x_2 \end{pmatrix},$$

where $\mu = 0.012277471$ is the mass of the moon, while $\tilde{\mu} = 1 - \mu$ is the mass of the earth. For the following starting values the solution should be periodic:

$$\mathbf{x}(0) = \begin{pmatrix} 0.994 \\ 0 \end{pmatrix}, \quad \dot{\mathbf{x}}(0) = \begin{pmatrix} 0 \\ -2.00158510637908252240537862224 \end{pmatrix}.$$

The period is $T = 17.0652165601579625588917206249$.

Solve the equation using these values on the time intervall $[0, T]$ using the explicit Euler method and `rk4` with $N = 1000, 6000, 10000$. For `expeuler` even try $N = 24000$.

Hint. If you do not want to type the above numbers, you can copy them from this PDF.

(c) Finally, plot the result of MATLAB's `ode45`, and check how many steps were needed (`length(t)`). This function uses an adaptive step size strategy that will be explained later in the lecture. Plot the curves of $x_1, x_2, \dot{x}_1$, and $\dot{x}_2$ to get an idea why the other methods are not suited.