1 ▶ Tangent cone and linearized feasible directions

Consider again the tangent cone $T_{Ω}(x)$ from the last exercise sheet:

$$x = (-1, 0, 0)^T, \quad Ω = \{x ∈ R^3: x_1^2 + x_2^2 ≤ 1, x_3^2 = 0\}.$$

Sketch $F(x)$. Modify the parametrization of $Ω$ such that $T_{Ω}(x) = F(x)$ holds.

2 ▶ Lagrange multipliers

Show that the Lagrange multipliers $λ^*$ and $μ^*$ in Theorem 5.11 are unique.

3 ▶ KKT conditions

Find the point(s) on the parabola $y = \frac{1}{5}(x−1)^2$ which is/are closest to $(x, y) = (1, 2)$ in the Euclidian norm, i.e., solve

$$\min_{x,y} (x−1)^2 + (y−2)^2 \quad \text{subject to} \quad (x−1)^2 = 5y.$$

Find all KKT points, and the solution. Check that replacing $(x−1)^2$ in the objective function by $5y$ and solving the unconstrained problem for $y$ does not work.

4 ▶ KKT conditions

For a fixed parameter $t$ consider

$$\min_{x ∈ R^2} \left( x_1 - \frac{3}{4} \right)^2 + (x_2 − t)^4 \quad \text{subject to} \quad \begin{bmatrix} 1 - x_1 - x_2 \\ 1 - x_1 + x_2 \\ 1 + x_1 - x_2 \\ 1 + x_1 + x_2 \end{bmatrix} ≥ 0.$$

(a) For what values of $t$, if there are any, does $x = (1, 0)^T$ satisfy the KKT conditions?

(b) Show that when $t = 1$, only the first constraint is active at the solution, and find the solution.

5 ▶ Geometric and arithmetic mean

(a) Use the KKT conditions to find the global solution of

$$\min_{x ∈ R^n} \sum_{i=1}^{n} x_i \quad \text{subject to} \quad \prod_{i=1}^{n} x_i = 1, \ x ≥ 0.$$

Hint. First show that the global minimizer satisfies $x > 0$.

(b) Use (a) to prove the inequality between the geometric and the arithmetic mean:

$$\left( \prod_{i=1}^{n} x_i \right)^{1/n} ≤ \frac{1}{n} \sum_{i=1}^{n} x_i \quad \text{for all} \ x ∈ R^n \ \text{with} \ x ≥ 0.$$